# Manifolds of Classical Mechanics

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**Abstract**

Some people claim that Lagrangian Mechanics is not "geometrical," I disagree. In the mean time, many physists quote "configuration space" without a detailed geometrical description. Refutation to that point and establishment of the geometrical picture will be presented in this article. Certainly, I will not confine the focus on Lagrangian mechanics only, nevertheless, I will portrait the big picture of classical mechanics. And differential geometry is the language used in the drama where three important manifolds will be our leading roles. They are named as Euclid-Newtonian manifold, configuration space (Lagrangian manifold) and phase space (Hamiltonian manifold) respectively. Finally, we will have a unified landscape for classical mechanics.

## 1 Introduction

In modern vision, mechanics is geometry. However, this idea is not fresh at all, at the Genesis of physics, I mean Galileo Galilei and Isaac Newton's times, physics was indeed geometry, Euclidian geometry to be exact. If you have read Newton's *Principia* you may confirm this idea.

**A survey into Newton's achievements** Newton's greatest achievement must be the definition of force, that is $\vec{F} = m\vec{a} = m\ddot{\vec{r}}$. The first and second law are entailed in this formula. The third law is merely a natural assumption about forces, which is equivalent to the statement that if one object is not imposing a force on the other object there could not be a reacting force on the original object. The third one is trivial but without it, the formula could only hold for one object. Another important contribution of Newton is the inverse square law of gravitational force which impacted all later branches of physics.

**Refinements of Newton's ideas** "Force", a brilliant invention as it was, is inconvenient for today's theories. What we now concern are potential, energy, action etc. But the significance of Newton's formula $\vec{F} = m\ddot{\vec{r}}$ is that it pins down the "second-order-derivative" form of all mechanics[1]. On the other hand, Euclidian geometry is the only version available in his times. Therefore, Newton's theories certainly fit well into Euclidian space i.e. $\mathbb{R}^n$.

In differential geometry, $\mathbb{R}^n$ is called the "trivial" manifold since manifolds are generalizations of "flat" spaces. Now let's begin our first construction with $\mathbb{R}^n$.

[1]Of course, mechanics could be of higher orders but this important topic is obviously worth a full long discussion. Hence, the coverage of higher order mechanics is not presented in this article.

**Definition: 1.1** (Euclid-Newtonian manifold)**.** For a $n$-particle system, the most general Euclid-Newtonian manifold takes the form

$$\mathcal{N} := \underbrace{\mathbb{R}^3 \times \mathbb{R}^3 \times \ \ldots \times \mathbb{R}^3}_{n} = \mathbb{R}^{3n}$$

where each $\mathbb{R}^3$ is the ordinary 3 dimensional space and since there are $n$ of them, we combine them into one object using Cartesian products.

Then I introduce vectors on $\mathcal{N}$.

**Proposition: 1.1.** *Let* $\mathbf{x}$ *denote the coordinates of* $\mathcal{N}$*, then*

$$\dot{\mathbf{x}} \equiv \frac{d\mathbf{x}}{dt}$$

*is the components of tangent vector* $d/dt$ *on* $\mathcal{N}$ *where* $t$ *is time.*

*Proof.* By definition of tangent vectors and Leibnitz's rule,

$$\frac{d}{dt} \equiv \frac{d\mathbf{x}}{dt}\frac{\partial}{\partial \mathbf{x}} = \dot{x}^i \partial_i$$

where Einstein summation convention is utilized. □

## 2 Construction of configuration space

Now that Euclid-Newtonian manifold is at our disposal, we then are able to construct configuration space but first let's work on constraints. To make things neat, all of our constraints are independent of time.

Constraints are often introduced as a scalar function of all the coordinates equal to a constant[2] which is written as $f(\mathbf{x}) = 0$. If I differentiate or, take the gradient of $f$, the equation reads

$$df = \partial_i f dx^i.$$

Further more, if I concentrate on a vicinity of $\mathbf{x}_0$, the differential $dx^i$ can be replaced by a finite difference $\Delta x^i$, hence, it becomes a linear algebraic equation which is

$$\Delta f(x_0^i) = \partial_i f(x_0^i)\Delta x^i.$$

Moreover, if there are more than one constraints i.e. $\{f^{(l)}\}_{l=1}^r$ where $r < 3n$, then we are able to write

$$\Delta f^l = F_i^l \Delta x^i$$

where $F_i^l := \partial_i f^l(x_0^i)$, a $r \times 3n$ matrix. Obviously, $F$ is singular since $r < 3n$ but if we reduce $s = 3n - r$ variables from $\Delta x$ and $s$ columns from $F$ keeping the remaining rows independent, then we have a reduced invertible matrix $G$. The matrix equation transforms into

$$\Delta f^l = G_j^l \Delta x^j$$

where $j = 1, 2, \ldots, r$. To massage it further,

$$\Delta x = G^{-1}\Delta f$$

[2]For convenience but without loss of any generality, the RHS is set to zero

which means you can read off $x^j$ directly from the given scalar functions. In another word, there are only $s$ coordinates unknown in the vicinity of $\mathbf{x}_0$, or they are "free" coordinates. That is the reason why $s$ is called the "degree of freedom". Actually the formula $s = 3n - r$ lies at the heart of linear algebra where $r$ is the rank of the matrix and $s$, the rank of null space. The formula demonstrates the fact that there are only $s$ basis vectors living on the constrained space and thus, the dimension is $s$. If I sew all the coordinate patches together, I have a manifold of dimension $s$. Our discussions lead to the following proposition.

**Proposition: 2.1.** *For a $3n$ dimensional Euclid-Newtonian manifold $\mathcal{N}$, a constraint is equivalent to a hypersurface of dimension $3n-1$. We denote such an object as $c^1_{\mathcal{N}}$. And if there are $r$ of them, we define*

$$\mathcal{C} := \bigcap_{l=1}^{r} c^l_{\mathcal{N}}$$

*which is of dimensionality $s = 3n - r$. $\mathcal{C}$ is the* configuration space *or the* Lagrangian manifold.

Now I define the independence of the constraints.

**Definition: 2.1.** $c^l_{\mathcal{N}}$ are said to be independent if the gradient of $f^{(l)}$ are linearly independent.

This definition corresponds to the independence of the rows of $F^l_i$.

## 3 From Lagrangian to Hamiltonian manifold

First, let's derive Euler-Lagrangian equation by doing variations on Lagrangian manifold $\mathcal{C}$. For theoretical interests, we assume the constraints are ideal, i.e. the virtual work of constraint forces always vanishes.

**Theorem: 3.1.** *Let $q^i$ denote the $s$ coordinates of $\mathcal{C}$. If I require variation $\delta q^i$ vanish at both ends of a curve $\gamma$, then the path integral of variations of $\delta L + Q_i \delta q^i$ is zero where $Q$ is the generalized force and $L = T - V$, kinetic energy subtracts potential energy.*

*Proof.*

$$\begin{aligned}
\delta \int_\gamma dt\, T &= \int dt\, \frac{1}{2} m \delta(\dot{\mathbf{x}}^2) \\
&= \int dt\, m\dot{\mathbf{x}} \cdot \delta\dot{\mathbf{x}} = \int dt\, m\dot{\mathbf{x}} \cdot \delta \frac{d\mathbf{x}}{dt} \\
&= 0 - \int dt\, m\ddot{\mathbf{x}} \delta\mathbf{x} = - \int dt\, \mathbf{F} \cdot \delta\mathbf{x} \\
&= - \int dt\, Q_i \delta q^i
\end{aligned}$$

If some of the $Q_i$'s are conservative i.e. $Q_j = -\frac{\partial V}{\partial q^j}$, and $\partial_j V \delta q^i \equiv \delta V$, then we can rewrite the integral as

$$\int dt\, \delta(T - V) = - \int dt\, Q_k \delta q^k.$$

Note that the left hand side is just $\delta L$, then

$$\int_\gamma dt\, \delta L + Q^k \delta q_k = 0.$$

□

If one manipulates $\delta L$, he may find this

$$\int_\gamma dt \left[\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} - Q_i\right] \delta q^i = 0$$

and conclude that

$$\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}^i}\right) - \frac{\partial L}{\partial q^i} - Q_i = 0$$

since the integral is true for all $\delta q^i$'s. Thus we find the Euler-Lagrangian equation by applying variational calculus on configuration manifold.

Now if $Q_k$'s are not present, then the theorem says "the variational gradient ($\delta$ operator) of scalar field $L$ adds up to zero along the integral curve". Note that

$$\delta L = \frac{\partial L}{\partial q^i}\delta q^i + \frac{\partial L}{\partial \dot{q}^i}\delta \dot{q}^i$$

which means the "delfferentiation" operator $\delta$ takes generalized velocities also as coordinates. But $\dot{q}^i$ is the component of the tangent vector at $q^i$, I have already shown that in section 1, and the proof is still valid here.

**Proposition: 3.1.** *The $\delta$ operator is defined on the tangent bundle of $\mathcal{C}$, which is denoted as $\mathcal{T}(\mathcal{C})$.*

**Proposition: 3.2.** *Lagrangian $L : \mathcal{T}(\mathcal{C}) \to \mathbb{R}$ is a scalar function defined on $\mathcal{T}(\mathcal{C})$ with coordinates $(q^i, \dot{q}^i)$.*

Now it is proper to introduce Hamiltonian mechanics.

**Proposition: 3.3.** *The manifold of Hamiltonian mechanics or the phase space, is $\mathcal{T}^*(\mathcal{C})$ with coordinates $(q^i, p_i)$ on which the Hamiltonian $H : \mathcal{T}^*(\mathcal{C}) \to \mathbb{R}$ is defined.*

Hamiltonian is obtained from Lagrangian by a Legendre transformation i.e.

$$H = p_i\dot{q}^i - L.$$

The product of momenta and velocities can be thought of as a pairing of an element of the tangent space with its dual. Suppose vector $\dot{q} \in \mathcal{T}_\mathcal{C}$ and 1-form $p \in \mathcal{T}^*_\mathcal{C}$, then

$$p[\dot{q}] = p_i dq^i[\dot{q}^j\partial_j] = \dot{q}^i p_j(dq^i\partial_j)$$

and for the fact that

$$df[\frac{d}{d\lambda}] \equiv \frac{df}{d\lambda}$$

then

$$p[\dot{q}] = p_i\dot{q}^j\delta^i_j = p_i\dot{q}^i.$$

The effect of Legendre transformation is to replace $\dot{q}^i$ by $p_i$ as the second set of independent variables or replace $\mathcal{T}(\mathcal{C})$ by $\mathcal{T}^*(\mathcal{C})$. More discussions about Hamiltonian mechanics will definitely lead to the theory of symplectic geometry. For a rigorous discussion, see the reference. Here I quote some results. The symplectic 2-form is

$$\omega = dq^i \wedge dp_i.$$

The Hamiltonian vector field is defined as

$$\mathbf{X}_H := \omega^\sharp(dH) \equiv (dH)^\sharp$$

and the triplet $(\mathcal{C}, \omega, \mathbf{X}_H)$ is called the Hamiltonian system. Then the trajectory of time evolution, or history of a mechanical system, is an integral curve of the Hamiltonian vector field.

## References

[1] Sadri Hassani (1998) Mathematical Physics—A Modern Introduction to Its Foundations, Springer