The Language of General Relativity
Essential Knowledge to Get Started

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Abstract

There are two reasons why beginners complain that general relativity is difficult. One is that the geometrical picture is hard to visualize: 4 dimensional, curved spacetime. The other one is the startling tensor algebra. Frankly speaking, nobody could visualize 4-D curved space without the help of logics and analogies which indeed take a lot of talents. Nevertheless, once you conquered the algebra, the pain would be much eased. This article is a summary of my knowledge of the algebra.

1 Introduction

What's general relativity all about? Well, it could be summed up in two statements:

- Spacetime is a curved pseudo-Riemannian manifold with a metric of signature $(-+++)$.
- The relationship between matter and the curvature of spacetime is contained in the equation

$$R_{\mu\nu} - \frac{1}{2}Rg_{\mu\nu} = 8\pi G T_{\mu\nu}.$$

It is helpful to introduce manifolds since tensors fields are defined on them. An n-manifolds are a class of geometrical objects that locally look like[1] $\mathbb{R}^n$. The second statement could be decoded as "matter curves spacetime while spacetime tells matter where to go." Next, let's run into tensors. I will first provide general ideas of tensors then move on to specific ones.

[1] Homomorphism, to be precise.

2 Tension of tensors

2.1 Vectors and 1-forms

Definition: 2.1. Let M be a differential manifold. A tangent vector at $P \in M$ is an operator $\mathbf{t}$: $F^{\infty} \to \mathbb{R}$ such that for every $f, g \in F^{\infty}$ and real numbers α, β

- $\mathbf{t}$ is linear: $\mathbf{t}(\alpha f + \beta g) = \alpha \mathbf{t}(f) + \beta \mathbf{t}(g)$;
- $\mathbf{t}$ satisfies the derivation property: $\mathbf{t}(fg) = g(P)\mathbf{t}(f) + f(P)\mathbf{g}$.

The vector is an abstraction of the derivative operator. A vector field is such tangent vectors defined at every point of the manifold. We often denote a vector as $d/d\lambda$, where $\lambda \in \mathbb{R}$ which parametrizes a smooth curve. In component notation, we write an upper index after the corresponding letter i.e. V^{μ}. 1-forms follows similar definitions, the only difference is to replace F^{∞} by V, the tangent vectors. In another word, 1-forms annihilate vectors into real numbers. It is denoted as ω_{ν}, a letter with a lower index.

2.2 General tensors

A general tensor is a multilinear map while a vector space is a linear map, of course. A (r, s)-tensor is defined as $T := V^1 \otimes V^2 \otimes \ldots \otimes V^r \otimes \omega_1 \otimes \omega_2 \otimes \ldots \otimes \omega_s$ where V^i are vectors and ω_j 1-forms. Each vector and 1-form comes from an individual linear space, which follows the definitions, and thus, the tensor made out of them is "multilinear". In a neat fashion to put it, a tensor is the tensor products of vectors and 1-forms. Thus, it denoted as $T^{\mu_1\mu_2\ldots\mu_r}{}_{\nu_1\nu_2\ldots\nu_s}$.

2.3 Form algebra

Since we have 1-forms, we are able to compose n-forms with the help of "$\otimes$". But actually, it is not the case. Forms are tensors with antisymmetric lower indices, therefore we use wedges ($\wedge$) to connect forms rather than $\otimes$.

Wedge product of p-form A and q-form B is

$$(A \wedge B)_{\mu_p\nu_q} \equiv (A \wedge B)_{\mu_1\ldots\mu_p\nu_1\ldots\nu_q} := \frac{(p+q)!}{p!q!} A_{[\mu_i\mu_p} B_{\nu_j\nu_q]}$$

where μ_i, ν_j abbreviate indices before μ_p, ν_q.

Properties of wedge products:

$$A \wedge B = (-1)^{pq} B \wedge A$$

$$(A \wedge B \wedge C)_{\mu_p\nu_q\rho_r} = \frac{(p+q+r)!}{p!q!r!} A_{[\mu_i\mu_p} B_{\nu_j\nu_q} C_{\rho_k\rho_r]}$$

Exterior derivatives The exterior derivatives are carried out by the d operator which has the following properties.

- Scalar: $df := \partial_\mu dx^\mu$ or $(df) := \partial_\mu f$
- Vector: $df\left(\frac{d}{d\lambda}\right) := \frac{df}{d\lambda}$
- Form: $d\omega := d(\omega_{\mu_i\mu_p} dx^{\mu_i} \wedge dx^{\mu_p}) = d\omega_{\mu_i\mu_p} \wedge dx^{\mu_i} \wedge dx^{\mu_p} = \partial_\lambda \omega_{\mu_i\mu_p} dx^\lambda \wedge dx^{\mu_i} \wedge dx^{\mu_p}$
  Or[2], $(d\omega)_{\lambda\mu_i\mu_p} = \partial_{[\lambda}\omega_{\mu_i\mu_p]} = \partial_\lambda \omega_{\mu_i\mu_p} - \partial_{\mu_i}\omega_{\lambda\mu_p} - \partial_{\mu_p}\omega_{\mu_i\lambda}$

Properties of wedge product Since d is nothing but the differential when acting on a scalar function, then it should satisfy the product rule. And indeed, it satisfies a modified product rule:

$$d(\omega \wedge \eta) = d\omega \wedge \eta + (-1)^p \omega \wedge d\eta$$

where ω is a p-form. If $\omega = dA$ then ω is said to be exact; if $d\omega = 0$, ω is closed. Then we have the following properties:

- If ω is exact then it is also closed. Or, $d \circ d \equiv 0$, since ∂'s commute;
- If the topology is trivial then closed forms are exact.

Hodge Duality In n dimensional space, we have the duality operation defined below.

$$(*A)_{\mu_i\mu_{n-p}} := \frac{1}{p!}\epsilon^{\nu_i\nu_p}{}_{\mu_i\mu_{n-p}} A_{\nu_i\nu_p}$$

$$**A = (-1)^{s+p(n-p)}A$$

3 Tensor acquaintances

3.1 The metric

The metric is the most fundamental quantity in general relativity which entails everything. The metric of our spacetime is a symmetric tensor with 2 lower indices. Let's a few examples.

- 3-D Euclidian space in Cartesian coordinates: $ds^2 := dx^2 + dy^2 + dz^2 := \delta_{ij} dx^i dx^j$
- 3-D Euclidian space in spherical coordinates: $ds^2 := dr^2 + r^2 d\theta^2 + r^2 \sin^2\theta d\varphi^2$
- 4-D flat spacetime: $ds^2 = -dt^2 + d\mathbf{x}^2 := \eta_{\mu\nu} dx^\mu dx^\nu$

[2] This last relation is true and pretty pragmatic.

3.2 Levi-Civita tensor—the volume element

First introduce the Levi-Civita symbol:

$$\widetilde{\epsilon}_{\mu_i \mu_n} = \begin{cases} +1, & \text{even permutation of } 01\ldots(n-1); \\ -1, & \text{odd permutation of } 01\ldots(n-1); \\ 0, & \text{not a permutation.} \end{cases}$$

For a matrix $M^{\mu}{}_{\mu'}$,

$$\tilde{\epsilon}_{\mu'_i \mu'_n} |M| = \tilde{\epsilon}_{\mu_i \mu_n} M^{\mu_i}{}_{\mu'_i} M^{\mu_n}{}_{\mu'_n}$$

3.3 Electromagnetic tensor

The electromagnetic tensor is defined in this fashion,

$$F_{\mu\nu} := \begin{pmatrix} 0 & -E_1 & -E_2 & -E_3 \\ E_1 & 0 & B_3 & -B_2 \\ E_2 & -B_3 & 0 & B_1 \\ E_3 & B_2 & -B_1 & 0 \end{pmatrix}$$

And Maxwell's equations in curved spacetime is

$$\nabla_\mu F^{\nu\mu} = 4\pi J^\nu$$
$$\nabla_{[\mu} F_{\nu\lambda]} = 0$$

or in form[3] notation

$$dF = 0 \qquad d(*F) = *J.$$

In flat spacetime, $F = dA$ where A is the 4-vector potential. There is an easy formula to generate $*F$. Write $\mathcal{E} = \mathbf{E} + i\,\mathbf{B}$, then $*\mathcal{E} = i(\mathbf{E} + i\,\mathbf{B}) = -\mathbf{B} + i\,\mathbf{E}$. the formula says the dual field is just $\mathbf{E} \to \mathbf{B}$ and $\mathbf{B} \to -\mathbf{E}$.

3.4 Curvatures

Riemannian tensor is defined as

$$R^\rho{}_{\sigma\mu\nu} := \partial_\mu \Gamma^\rho_{\nu\sigma} - \partial_\nu \Gamma^\rho_{\mu\sigma} + \Gamma^\rho_{\mu\lambda}\Gamma^\lambda_{\nu\sigma} - \Gamma^\rho_{\nu\lambda}\Gamma^\lambda_{\mu\sigma}$$

$$[\nabla_\mu, \nabla_\nu] V^\rho = R^\rho{}_{\sigma\mu\nu} V^\sigma - 2\,\Gamma^\lambda_{[\mu\nu]} \nabla_\lambda V^\rho$$

[3] $F_{\mu\nu}$ is a 2-form.

Properties of Riemann tensor

- $R_{\rho\sigma\mu\nu} = R_{[\rho\sigma]\mu\nu} = R_{\rho\sigma[\mu\nu]}$ (antisymmetric in first two and second two indices)
- $R_{\mu\nu\rho\sigma} = R_{\rho\sigma\mu\nu}$ (symmetric in first pair and second pair)
- $R_{\rho[\sigma\mu\nu]} = 0$ and $R_{[\rho\sigma\mu\nu]} = 0$
- $\nabla_{[\lambda} R_{\rho\sigma]\mu\nu} = 0$ (Bianchi Identity)

Other curvatures

- Ricci tensor is the trace of Riemann tensor, $R_{\mu\nu} = R^{\lambda}{}_{\mu\lambda\nu}$, which has the property that $R_{\mu\nu} = R_{\nu\mu}$.
- Ricci scalar $R = R^{\mu}{}_{\mu}$ (trace of Ricci tensor)
- Einstein tensor $G = R_{\mu\nu} - \frac{1}{2} R g_{\mu\nu}$, from Bianchi identity $\nabla^{\mu} G_{\mu\nu} = 0$

References

[1] Sadri Hassani: Mathematical Physics—A Modern Introduction to Its Foundations, Springer

[2] *Spacetime and Geometry* Sean M. Carroll, University of Chicago