Field-theoretic interpretations of Ginzburg-Landau theory of superconductivity

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Ginzburg-Landau theory of superconductivity is derived from microscopic BCS theory in terms of path integral formalism. Expression for the supercurrent is obtained from the Ginzburg-Landau action which is exactly the equation of motion of the massive photon field. It is further pointed out that the London penetration depth is actually the Compton wavelength of the massive photon.

I. INTRODUCTION

In 1950 Ginzburg and Landau proposed a phenomenological theory describing the superconductivity phase transition[3], however, Gor'kov showed that the microscopic Bardeen-Cooper-Schrieffer (BCS)[2] theory of superconductivity can derive the Ginzburg-Landau theory in a suitable limit[4].

In this paper, we will first write down the effective action from BCS Hamiltonian in terms of path integral formalism and then expand the action near to T_c in powers of order parameter. Then the concepts of off-diagonal long ranged order and spontaneous breaking of gauge symmetry will be explained and then Anderson-Higgs mechanism is pointed out to give a finite mass to photons inside the superconductor. New interpretations of Ginzburg-Landau equations and London penetration depth is shown to be directly related to the finite mass of photons. And a new picture for the mechanism of Josephson tunneling is provided on the basis of previous discussion. Through out the paper $\hbar = c = 1$.

II. DERIVATION OF GINZBURG-LANDAU ACTION

A. The BCS Hamiltonian

Phonons are known to mediate an effective attraction between electrons in a solid, namely the electron-phonon interaction, and this gives rise to a finite binding energy of a pair of electrons near the Fermi surface, however small the attraction is, which was demonstrated by Cooper[1], hence, the electron-phonon interaction finally leads to an overall instability of the Fermi sea. In the BCS theory, it is further assumed that the pairing electrons have opposite momenta and spins whose energies lie within a thin shell of width $\pm\hbar\omega_D$ at around the Fermi surface, where ω_D is the Debye frequency of the solid.[2] Finally, we have the BCS Hamiltonian

$$\mathscr{H} = \sum_{k,\sigma}(\varepsilon_k - \mu)c^\dagger_{k,\sigma}c_{k,\sigma} - V\sum_{k,k'}c^\dagger_{k,\uparrow}c^\dagger_{-k,\downarrow}c_{-k',\uparrow}c_{k',\downarrow} \tag{1}$$

where $V > 0$ is the effective attraction.

For later convenience, we define

$$C^\dagger = \sum_k c^\dagger_{k,\uparrow}c^\dagger_{-k,\downarrow} \tag{2a}$$

$$C = \sum_k c_{-k,\uparrow}c_{k,\downarrow} \tag{2b}$$

and the abbreviated BCS Hamiltonian reads

$$\mathscr{H} = \sum_{k,\sigma}\xi_k c^\dagger_{k,\sigma}c_{k,\sigma} - VC^\dagger C \tag{2c}$$

where $\xi_k = \varepsilon_k - \mu$.

The corresponding partition function of BCS Hamiltonian is given by the path integral

$$\mathscr{Z} = \int \mathscr{D}[\bar{c},c]\mathrm{e}^{-\int_0^\beta d\tau \sum_{k,\sigma}\bar{c}_{k,\sigma}(\partial_\tau+\xi_k)c_{k,\sigma} - V\bar{C}(\bar{c})C(c)} \tag{3}$$

We will remove the quartic interaction term by means of Hubbard-Stratonovich transformation.

B. Hubbard-Stratonovich transformation

We first introduce a complex pairing field $\Delta(\tau,x) = \rho(\tau,x)\mathrm{e}^{\mathrm{i}\theta(\tau,x)}$ and note the identity

$$1 = \int \mathscr{D}[\bar{\Delta},\Delta]\mathrm{e}^{-\int_0^\beta d\tau\int dx \bar{\Delta}(\tau,x)V^{-1}\Delta(\tau,x)} \tag{4}$$

Now we shift the integrand by a constant amount and it will not change the value of the integral, that is

$$1 = \int \mathscr{D}[\bar{\Delta},\Delta]\mathrm{e}^{-\int_0^\beta d\tau\int dx(\bar{\Delta}(\tau,x)-V\bar{C})V^{-1}(\Delta(\tau,x)-VC)} \tag{5}$$

$$= \int \mathscr{D}[\bar{\Delta},\Delta]\mathrm{e}^{-\int_0^\beta d\tau\int dx\, \bar{\Delta}V^{-1}\Delta + V\bar{C}C - \bar{\Delta}C - \bar{C}\Delta} \tag{6}$$

So we insert the identity into the path integral (3), we have

$$\mathscr{Z} = \int \mathscr{D}[\bar{c},c]\mathscr{D}[\bar{\Delta},\Delta]\mathrm{e}^{-S} \tag{7}$$

with the action defined as follows

$$S = \int_0^\beta d\tau \sum_{k,\sigma}\bar{c}_{k,\sigma}(\partial_\tau + \xi_k)c_{k,\sigma} + \frac{\bar{\Delta}\Delta}{V} - \bar{\Delta}C - \bar{C}\Delta \tag{8}$$

and note that the action is quadratic in all fermionic variables.

If we introduce Nambu notation, the action takes a more compact form,

$$S = \int_0^\beta d\tau \sum_k \bar{\psi}_k(\partial_\tau + h_k)\psi_k + \frac{\bar{\Delta}\Delta}{V} \tag{9a}$$

where

$$\psi_k = (c_{k,\uparrow}, \bar{c}_{-k,\downarrow})^T \tag{9b}$$

$$h_k = \begin{pmatrix} \xi_k & \Delta \\ \bar{\Delta} & -\xi_k \end{pmatrix} \tag{9c}$$

C. The effective action

Next we integrate out the fermion field in (9a) and obtain the effective action,

$$\mathscr{Z} = \int \mathscr{D}[\bar{\Delta},\Delta] e^{-S_{\text{eff}}[\bar{\Delta},\Delta]} \tag{10a}$$

$$S_{\text{eff}}[\bar{\Delta},\Delta] = \int_0^\beta d\tau \, \frac{\bar{\Delta}\Delta}{V} + \sum_k \text{Tr}\ln(\partial_\tau + h_k) \tag{10b}$$

Making saddle-point approximation, we obtain the partition function and free energy,

$$\mathscr{Z} = \exp(-S_{\text{eff}}) \tag{11a}$$

$$F = -\ln(\mathscr{Z})/\beta = S_{\text{eff}}/\beta \tag{11b}$$

The free energy is directly related to the effective action by a factor of β. We recover the Ginzburg-Landau theory by expanding the effective action in terms of Δ and Fourier transforming it into real space,

$$S_{\text{eff}}/\beta = \int dx \left\{ \frac{1}{4m}|\partial\Delta|^2 + V + \mathscr{O}(|\Delta|^6) \right\} \tag{12a}$$

$$V = -a|\Delta|^2 + \frac{b}{2}|\Delta|^4 \tag{12b}$$

where $a, b \geq 0$.

III. ELECTRODYNAMICS OF A SUPERCONDUCTOR

A. Anderson-Higgs mechanism

Requiring the effective theory to possess local gauge invariance, it is necessary to introduce the gauge field $A_\mu = (\phi, A)$, namely replacing partial derivative by covariant derivative, and together with $U(1)$ gauge transformation

$$D_\mu = \partial_\mu + 2\text{i}eA_\mu \tag{13a}$$

$$\Delta \to \Delta e^{\text{i}\alpha(x)} \tag{13b}$$

$$A_\mu \to A_\mu - \frac{1}{2e}\partial_\mu \alpha(x) \tag{13c}$$

Now we decompose the order parameter into amplitude and phase $\Delta = \rho e^{\text{i}\theta}$, eqn (12a) becomes

$$S_{\text{eff}}/\beta = \int dx \left\{ \frac{1}{4m}|D_\mu(\rho e^{\text{i}\theta})|^2 + V(\rho) \right\} \tag{14a}$$

$$V(\rho) = -a\rho^2 + \frac{b}{2}\rho^4 \tag{14b}$$

Expand eqn (14a) and add in the kinetic part of Maxwell field $\frac{1}{4}F^{\mu\nu}F_{\mu\nu}$, we have

$$S_{\text{eff}}/\beta = \int dx \left\{ \frac{1}{4m}(\partial_\mu\rho)^2 + \frac{(2e\rho)^2}{4m}(\frac{1}{2e}\partial_\mu\theta + A_\mu)^2 + V(\rho) + \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \right\} \tag{15}$$

$$= \int dx \left\{ \frac{1}{4m}(\partial_\mu\rho)^2 + \frac{(e\rho)^2}{m}(\frac{1}{2e}\partial_\mu\theta + A_\mu)^2 + V(\rho) + \frac{1}{4}F^{\mu\nu}F_{\mu\nu} \right\} \tag{16}$$

$$= \int dx \left\{ \frac{1}{4m}(\partial_\mu\rho)^2 + \frac{m_A^2}{2}\tilde{A}_\mu^2 + V(\rho) + \frac{1}{4}\tilde{F}^{\mu\nu}\tilde{F}_{\mu\nu} \right\} \tag{17}$$

In the last line a gauge transformation is performed such that the θ-field is 'eaten up', where $m_A^2 = 2e^2|\Delta|^2/m$.

The m_A in eqn (17) is twice square root of the superfluid phase stiffness, that is, the energy cost by the gradient of θ and it also serves as the mass of Maxwell field. The total gradient cost of bulk superconductor is of order $\mathscr{O}(V)$ which in thermodynamic limit, equals infinity, therefore, the system exhibits off-diagonal long ranged order by spontaneously breaking the $U(1)$ gauge symmetry. According to Goldstone's theorem, there should be gapless excitation similar to spinons in a magnet but it is 'eaten up' by the gauge field and becomes the longitudinal mode of photon field bestowing a mass m_A to photons.

B. Ginzburg-Landau equations

Equations of motion can be obtained from the action by means of functional derivation. Before that, we rewrite the action in a more compact form

$$S_{\text{eff}}/\beta = \int dx \left\{ \frac{1}{4m}(\text{d}\rho)^2 + \frac{m_A^2}{2}\left(\frac{\text{d}\theta}{2e} + A\right)^2 + V(\rho) + \frac{1}{2}(\text{d}\wedge A)^2 \right\} \tag{18}$$

Take functional derivative with respect to A, we have

$$0 = \frac{\delta S_{\mathrm{eff}}}{\beta \delta A} = -m_A^2 \left(\frac{\mathrm{d}\theta}{2e} + A \right) - \mathrm{d} \wedge \mathrm{d} \wedge A \tag{19}$$

Then we have the Ginzburg-Landau equation for supercurrent[3]

$$J = \mathrm{d} \wedge \mathrm{d} \wedge A = -m_A^2 \left(\frac{\mathrm{d}\theta}{2e} + A \right) \tag{20}$$

which is exactly the equation of motion for massive photon field.

For the ground state, θ is uniform, then we have the London version of supercurrent[6]

$$J = -m_A^2 A = -\frac{|\Delta|^2 e^2}{m} A \tag{21}$$

and together with two simultaneous equations of motion for vector potential magnetic field,

$$\nabla \times \nabla \times A + m_A^2 A = 0 \tag{22a}$$
$$\nabla \times \nabla \times B + m_A^2 B = 0 \tag{22b}$$

whose solution is accountable for the celebrated Meissner effect[5].

Another London equation[6] is trivially obtained by noting

$$E = \partial_t A - \nabla \phi = -1/m_A^2 \dot{J} \tag{23}$$

C. The massive photons

Choosing London gauge, namely $\nabla \cdot A = 0$, and by the fact that $\nabla \cdot B \equiv 0$, eqn (22a) and (22b) are reduced to the following

$$\nabla^2 A = m_A^2 A, \quad \nabla^2 B = m_A^2 B \tag{24}$$

The depth of penetration is characterized by the length scale $\lambda_L = 1/m_A$; the London penetration depth is directly related to the mass of photon which means the damping of external magnetic field is totally caused by the finite mass of photon inside the superconductor.

The explanation is as follows: for a massless photon, its propagator falls off as $1/r$ but when photon obtains a finite mass, the propagator drops as $\exp(-m_A r)/r$ where $1/m_A$ is the Compton wavelength for the massive photon. Photon from outside cannot penetrate further than the length scale $1/m_A$ since the massive electromagnetic field is short ranged.

The supercurrent $J \propto A$, is only maintained in the skin of the bulk superconductor since it is a consequence of equation of motion for the photons which are limited by its finite force range. Symmetry of photon polarization state is not broken since there is no 'Zeeman-like' term in the action, therefore, polarizations of external photons are preserved namely, only right or left (± 1) polarization states; phase fluctuations well inside the superconductor or the Goldstone modes must be photons in the zero polarization state since photons carrying the other two polarizations are prohibited from outside. In the ground state where phase is uniform, there is no anomalously polarized photon. However, in a Josephson junction, perturbed phases propagate in opposite directions ($\pm \exp(\pm \delta\theta)$) which when summed together gives a $\sin(\delta\theta)$-shape supercurrent.

IV. CONCLUSIONS

We start from the microscopic BCS theory of superconductivity and recover the Ginzburg-Landau theory in the neighborhood of T_c by expanding the effective action, obtained from path integral formalism, up to $|\Delta|^4$. Then Anderson-Higgs mechanism was demonstrated to give rise to a finite mass of photon. Next we proceed to the consequences brought about by the effect of photon mass. First the formula for the supercurrent is actually the equation of motion for the massive photon field. Second the London penetration depth is pointed out to be the Compton wavelength of the massive photons. The finite mass also makes the anomalous zero polarization state possible, but it is pointed out that it cannot be observed in the ground state whereas such anomalous photons are responsible for the Josephson current.

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[1] L. N. Cooper, Phys. Rev. **104**, 1189 (1956)
[2] Bardeen, Cooper, and Schrieffer, Phys. Rev. **108**, 1175 (1957)
[3] V.L. Ginzburg and L.D. Landau. Zh. Eksp. Teor. Fiz., **20** (1950), p. 1064.
[4] LP Gor'kov J. Exptl. Theoret. Phys. (U.S.S.R.) **36**, 1918-1923 (June, 1959)
[5] W. Meissner and R. Ochsenfeld, Naturwiss. **21**, 787 (1933).
[6] H. London and F. London, Proc. Roy. Soc. (London) **A149**, 71 (1935); Physica **2**, 341 (1935)